

Geometric measure of quantum discord and the geometry of a class of two-qubit states

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We investigate the geometric picture of the level surface of quantum entanglement and geometric measure of quantum discord (GMQD) of a class of X states, respectively. This pictorial approach provides us a direct understanding of the structure of entanglement and GMQD. The dynamic evolution of GMQD under two typical kinds of quantum decoherence channels is also investigated. It is shown that there exist a class of initial states for which the GMQD is not destroyed by decoherence in a finite time interval. Furthermore, we establish a factorization law between the initial and final GMQD which allows us to infer the evolution of discord under the influences of the environment.

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I. INTRODUCTION

Entanglement is regarded as an ingredient resource for performing almost all quantum information processing tasks[1]. The situation started to change until a computational model was presented named deterministic quantum computation with one qubit(DQC1)[2]. Quantum discord was considered to be the figure of merit for this model of quantum computation. Ever since, quantum discord has attracted much attention[3–27]. Quantum discord is a measure of nonclassical correlations between two subsystems of a quantum system. It quantifies how much a system can be disturbed when people observe it to gather information. Such quantum correlations may be present in separable states and have a non vanishing value for almost all quantum state[12]. On the other hand, the quantum discord can be used to indicate the quantum phase transitions better than entanglement in certain physical systems at finite temperature [28, 29]. In particular, two operational interpretations of quantum discord have been proposed, one in thermodynamics [30] and the other from the information theoretic perspective through the state merging protocol [31, 32]. These results establish the status of quantum discord as another important resource for quantum informational processing tasks besides entanglement.

Originally, the first definition of quantum discord is due to Ollivier and Zurek [3] and, independently, by Henderson and Vedral [4]. However, it is notoriously difficult to compute it because of the minimization taken over all possible POVM, or von Neumann measurements. At present, there are only a few analytical results including the Bell-diagonal states [5], rank-2 states [10, 33] and Gaussian state [34, 35]. In addition, a simple algorithm to evaluate the quantum discord for two-qubit X-states is proposed by Ali *et al.* [36] with minimization taken over only a few cases. Unfortunately, their algorithm is valid only for a family of X-states[8, 37]. Recently Shi *et al.* [38] present an efficient method to solve this

problem. For the general two-qubit states, the evaluation of quantum discord remains a nontrivial task and only some lower and upper bounds are available [39]. In order to avoid the difficulties in minimization procedures a geometric view of quantum discord was introduced. Generally, there are two version of geometric measure of quantum discord (GMQD). In the first version the concept of relative entropy is used as a distance measure of correlations [6]. The second version is defined by the Hilbert-Schmidt norm measure [7]. The relative-entropy-based discord have the drawback that their analytical expressions are known only for certain limited classes of states. Below we only consider the second version of geometric measure of quantum discord. Formerly, this geometric measure of quantum discord is defined by $D_A^g = \min_{\chi \in \Omega_0} \|\rho - \chi\|^2$, where Ω_0 denotes the set of zero-discord states and $\|X - Y\|^2 = \text{Tr}(X - Y)^2$ is the square norm in the Hilbert-Schmidt space. The subscript A denotes that the measurement is taken on the system A . An arbitrary two-qubit state can be written in Bloch representation:

$$\rho = \frac{1}{4} \left[I \otimes I + \sum_i^3 (x_i \sigma_i \otimes I + y_i I \otimes \sigma_i) + \sum_{i,j=1}^3 R_{ij} \sigma_i \otimes \sigma_j \right] \quad (1)$$

where $x_i = \text{Tr}(\rho(\sigma_i \otimes I))$, $y_i = \text{Tr}(\rho(I \otimes \sigma_i))$ are components of the local Bloch vectors, $\sigma_i, i \in \{1, 2, 3\}$ are the three Pauli matrices, and $R_{ij} = \text{Tr}(\rho(\sigma_i \otimes \sigma_j))$ are components of the correlation tensor. For two-qubit case, the zero-discord state is of the form $\chi = p_1 |\psi_1\rangle \langle \psi_1| \otimes \rho_1 + p_2 |\psi_2\rangle \langle \psi_2| \otimes \rho_2$, where $\{|\psi_1\rangle, |\psi_2\rangle\}$ is a single-qubit orthonormal basis. Then a analytic expression of the GMQD is given by [7]:

$$D_A^g(\rho) = \frac{1}{4} \left(\|x\|^2 + \|R\|^2 - k_{\max} \right) \quad (2)$$

where $x = (x_1, x_2, x_3)^T$ and k_{\max} is the largest eigenvalue of matrix $K = xx^T + RR^T$.

In this paper we shall investigate the level surfaces of entanglement and GMQD for a class of two-qubit states using the geometric picture presented in [40]. It is well known that the set of Bell-diagonal states for two qubits can be depicted as a tetrahedron in three dimensions in Bloch representation [41]. Analogous to entanglement, Lang and Caves have depicted the level surfaces of quantum discord for Bell-diagonal states. They found that the picture and the quantum discord are very different. This pictorial approach provides a complete and direct understanding of the behavior of von-Neumann measurement-based quantum discord. It has been shown that the quantum discord and GMQD do not necessarily imply the same ordering of two-qubit X-states [42–44]. More recently, Girolami and Adesso [45] and independently Battle *et al.* [46] provided a numerical evidences, from which one can infer that there exist other states violating the states ordering with quantum discords. It is thus worth investigating the GMQD from the geometric picture. In this sense, our research provides a direct understanding of the structure of GMQD at first time. We consider a class of X states that the Bloch vectors are z directional, which including Bell-diagonal states as a special case. We show that the level surface of GMQD is very different with the von-Neumann-measurement-based quantum discord. On the other hand, the dynamics of quantum discord has attracted much attention due to the inevitable interaction with environment. The dynamic behavior of GMQD is also investigated under two typical kinds of decoherence channels. We find a class of states for which the GMQD is not destroyed by decoherence in a finite time interval. Interestingly, we also obtain a factorization law for GMQD which allows us to infer the evolution of discord under the influences of the environment, *e.g.* the phase damping channel(PDC), and the depolarizing channel(DPC).

The paper is organized as follows. In Sec.II, we present the geometric picture of the level surface of quantum entanglement and GMQD of a class of X states with the Bloch vectors are z directional, respectively. It is shown that the surface of constant GMQD varies with the local Bloch vectors. In Sec.III, we investigate the dynamic evolution of GMQD under quantum decoherence channels, and obtain analytic results for two typical kinds of quantum decoherence channels. A summary is given in Sec.IV.

II. GEOMETRICAL PICTURE OF ENTANGLEMENT AND GMQD

For analytical simplicity, we consider the following two-qubit X states:

$$\rho = \frac{1}{4} \left[I \otimes I + \mathbf{r} \cdot \boldsymbol{\sigma} \otimes I + I \otimes \mathbf{s} \cdot \boldsymbol{\sigma} + \sum_{i=1}^3 c_i \sigma_i \otimes \sigma_i \right] \quad (3)$$

where we choose the Bloch vectors are z directional with $\mathbf{r} = (0, 0, r), \mathbf{s} = (0, 0, s)$. The GMQD can be calculated explicitly for this state, thus allowing us to get analytic results. If $\mathbf{r} = \mathbf{s} = 0$, ρ reduces to the two-qubit Bell-diagonal states. Horodecki have shown that Bell-diagonal states belongs to a tetrahedron with vertices $(1, -1, 1), (-1, 1, 1), (1, 1, -1),$ and $(-1, -1, -1)$ in the Bloch representation. From the positivity of the eigenvalues of ρ in Eq.(3), we have

$$\begin{aligned} 0 &\leq \frac{1}{4} \left(1 - \sqrt{r^2 - 2rs + s^2 + c_1^2 + 2c_1c_2 + c_2^2 - c_3} \right) \leq 1, \\ 0 &\leq \frac{1}{4} \left(1 + \sqrt{r^2 - 2rs + s^2 + c_1^2 + 2c_1c_2 + c_2^2 - c_3} \right) \leq 1, \\ 0 &\leq \frac{1}{4} \left(1 - \sqrt{r^2 + 2rs + s^2 + c_1^2 - 2c_1c_2 + c_2^2 + c_3} \right) \leq 1, \\ 0 &\leq \frac{1}{4} \left(1 + \sqrt{r^2 + 2rs + s^2 + c_1^2 - 2c_1c_2 + c_2^2 + c_3} \right) \leq 1 \end{aligned} \quad (4)$$

For fixed parameters r and s , the above inequalities becomes a three-parameter set, whose geometry can be depicted in the three dimensional correlation state space. In Fig.1 we plot the physical region with different r and s , respectively. Fig.1 shows that physical regions of the state ρ shrink with larger r and s . We plot in Fig.2 the level surfaces of constant concurrence with fixed r and s for three cases. Here, we choose concurrence to measure entanglement which is defined as $C = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where the λ_i are, in decreasing order, the square roots of the eigenvalues of the matrix $\rho \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y$ where ρ^* is the complex conjugate of ρ . As shown in Fig.2, the level surfaces of constant concurrence for the state ρ defined in Eq.(3) consist of four discrete pieces, and the areas decrease with larger concurrence. A extremal case is the four vertices $(1, -1, 1), (-1, 1, 1), (1, 1, -1),$ and $(-1, -1, -1)$ of the tetrahedron correspond to the four Bell states with maximal concurrence. Finally, we investigate the GMQD from the geometric picture. For the state ρ , the GMQD can be calculated in the method presented in Ref. [47]. By introducing a matrix \mathcal{R} defined by

$$\mathcal{R} = \begin{pmatrix} 1 & y^T \\ x & R \end{pmatrix} \quad (5)$$

and 3×4 matrix \mathcal{R}' through deleting the first row of \mathcal{R} . Then the GMQD is given by

$$D_A^g(\rho) = \frac{1}{4} \left[\left(\sum_k \lambda_k^2 \right) - \max_k \lambda_k^2 \right] \quad (6)$$

where λ_k is the singular values of \mathcal{R}' . For the two-qubit state ρ in Eq.(3), we obtain

$$D_A^g(\rho) = \frac{1}{4} (c_1^2 + c_2^2 + c_3^2 + r^2 - \text{Max}(c_1^2, c_2^2, c_3^2 + r^2)) \quad (7)$$

The geometric picture is depicted in terms of the constant GMQD in Fig.3. From these plots one can see that the shape of the constant GMQD is quite different from von-Neumann-measurement-based quantum discord. It also shows different shape for different local Bloch vectors. The constant surfaces are cut of by the physical region of state ρ . For small discord the surface is continuous, and it becomes discrete pieces for larger discord. At the four vertices $(1, -1, 1)$, $(-1, 1, 1)$, $(1, 1, -1)$, and $(-1, -1, -1)$ of the tetrahedron the GMQD reach its maximal value. Furthermore, one can see that GMQD is neither concave nor convex as shown in Fig.3.

III. DYNAMICS OF GMQD UNDER LOCAL DECOHERENCE CHANNELS

In this section we consider the state is affected by the action of two independent channels and calculate the GMQD analytically. The dynamics of quantum discord has been investigated in both Markovian and non-Markovian environments and has been demonstrated experimentally [48–56]. It has been shown that the behaviors of von-Neumann-measurement-based quantum discord and GMQD may be different, it is thus desirable to consider the evolution of GMQD under different decoherence channels. Here, we consider two typical kinds of decoherence channels: the phase damping channel(PDC), and the depolarizing channel(DPC). To calculate the dynamics of GMQD, we turn to the Heisenberg picture to describe the quantum channels. In order to obtain the analytic expressions of GMQD of the state subject to local decoherence channels, we need to calculate the expectation matrix \mathcal{R} . In the Heisenberg picture [47, 57], the expectation matrix \mathcal{R} is given by

$$\mathcal{R}_{ij} = (M_A \mathcal{R}_0 M_B^T)_{ij} \quad (8)$$

where $\mathcal{R}_0 = \text{Tr}(\sigma_i \otimes \sigma_j \rho_0)$, $i \in \{0, 1, 2, 3\}$, ρ_0 is the initial state, and $M_{A(B)}$ is the transmission matrix of each local channel. For simplicity, we choose the local channels to be identical. In this case, the transmission matrix can be written as

$$M_{PDC} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-p & 0 & 0 \\ 0 & 0 & 1-p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (9)$$

$$M_{DPC} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-p & 0 & 0 \\ 0 & 0 & 1-p & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}, \quad (10)$$

For state (3), $\mathcal{R}_0 = \{\{1, 0, 0, s\}, \{0, c_1, 0, 0\}, \{0, 0, c_2, 0\}, \{r, 0, 0, c_3\}\}$. According to the above formula, we have

$$D_{PDC}^g = \frac{1}{4} \left[(1-p)^4 c_1^2 + (1-p)^4 c_2^2 + r^2 + c_3^2 - \max \left\{ (1-p)^4 c_1^2, (1-p)^4 c_2^2, r^2 + c_3^2 \right\} \right] \quad (11)$$

$$D_{DPC}^g = \frac{1}{4} \left[(1-p)^4 c_1^2 + (1-p)^4 c_2^2 + (1-p)^2 r^2 + (1-p)^4 c_3^2 - \max \left\{ (1-p)^4 c_1^2, (1-p)^4 c_2^2, (1-p)^2 r^2 + (1-p)^4 c_3^2 \right\} \right] \quad (12)$$

For some Bell-diagonal states, it has been shown that von-Neumann-measurement-based quantum discord is not destroyed by decoherence for some finite time interval [51]. A natural question arises: Whether such phenomena exists for GMQD? We consider the state in Eq.(3) undergoes two identical PDCs. In this case, $p = 1 - \exp(-\gamma t)$, where γ is the phase damping rate. For $c_1 = 0, c_2^2 > r^2 + c_3^2$. Suppose p_1 satisfy the equation $(1-p_1)^4 c_2^2 = r^2 + c_3^2$. If $p < p_1$, from Eq.(11) we have $D_{PDC}^g = \frac{1}{4} (r^2 + c_3^2)$ which is independent of time. Therefore, we conclude that for a finite time interval the GMQD does not decay despite the presence of local phase damping noises. It is directly to see that such phenomenon also exists for the case $c_2 = 0, c_1^2 > r^2 + c_3^2$. These results show that GMQD remain intact under the action of some special kinds of quantum channels. It should be noted similar phenomenon have also been noticed by Karpat *et al* [59] for the qubit-qutrit systems. In the geometric picture, this behavior corresponds to the state evolves along a straight line in the constant GMQD tube until it encounters another constant GMQD tube.

Hitherto, the time evolution of GMQD under PDC or DPC can be described by Eq.(11) and Eq.(12). Next, we want to derive a more general result on GMQD relating the initial and final state of GMQD. Inspired by the famous factorization Law for entanglement decay derived by Konrad *et al.* [58], we find a analogous factorization law between the initial and final GMQD of the class of two-qubit states defined in Eq.(3) subject to two different local decoherence channels. We generalize our result as the following theorem.

Theorem. Consider the class of X states defined in Eq.(3), with each qubit being subject to the local decoherence channels, *i.e.* the phase damping channel(PDC) or the depolarizing channel(DPC). The time evolution of GMQD satisfies

$$D^g [(\$1 \otimes \$2) \rho(t)] \geq 2D^g [(\$1 \otimes \$2) |\beta_i\rangle \langle \beta_i|] D^g [\rho(0)] \quad (13)$$

where the local decoherence channels is represented by $\$1$ and $\$2$, ρ_0 is the initial state and $|\beta_i\rangle, i \in \{1, 2, 3, 4\}$ denotes one of the four Bell states.

Proof. First we consider the state $\rho(0)$ is effected by the action of two identical local PDCs, the time evolution of

GMQD is given by Eq.(11). For convenience we divide the proof into three cases.

Case 1. $r^2 + c_3^2 \geq \{c_1^2, c_2^2\}$. In this case, it is directly to show that the inequality becomes equality.

Case 2. $c_1^2 \geq c_2^2 \geq r^2 + c_3^2$. First, using Eq.(7) we have $D^g(\rho(0)) = \frac{1}{4}(c_2^2 + r^2 + c_3^2)$. Suppose p_0 satisfy the equation $(1 - p_0)^4 c_1^2 = r^2 + c_3^2$. If $p \leq p_0$, the GMQD of the state ρ is given by

$$\begin{aligned} D_{PDC}^g(\rho(0)) &= \frac{1}{4} [(1-p)^4 c_2^2 + r^2 + c_3^2] \\ &\geq \frac{1}{4} [(1-p)^4 c_2^2 + (1-p)^4 (r^2 + c_3^2)] \\ &= (1-p)^4 D^g(\rho(0)) \\ &= 2D^g[(\$_{PDC} \otimes \$_{PDC}) |\beta_i\rangle \langle \beta_i|] D^g(\rho(0)) \end{aligned} \quad (14)$$

If $p > p_0$, then

$$\begin{aligned} D_{PDC}^g(\rho(t)) &= \frac{1}{4} [(1-p)^4 c_1^2 + (1-p)^4 c_2^2] \\ &\geq \frac{1}{4} [(1-p)^4 c_2^2 + (1-p)^4 (r^2 + c_3^2)] \\ &= (1-p)^4 D^g(\rho(0)) \\ &= 2D^g[(\$_{PDC} \otimes \$_{PDC}) |\beta_i\rangle \langle \beta_i|] D^g(\rho(0)) \end{aligned} \quad (15)$$

Case 3. $c_1^2 \geq r^2 + c_3^2 \geq c_2^2$. For $p \leq p_0$ or $p > p_0$, the proof is similar to case 2.

By simply exchanging c_1 and c_2 we can verify the above relations also hold for the cases $c_2^2 \geq c_1^2 \geq r^2 + c_3^2$ and $c_2^2 \geq r^2 + c_3^2 \geq c_1^2$. For the case of two identical local DPCs, one can prove the above results by the same way as proving PDCs. In the following, we consider the first qubit is subject to the PDC and the second qubit is subject to the DPC. The expectation matrix \mathcal{R} can be calculated according to the formula $\mathcal{R} = M_A \mathcal{R}_0 M_B^T$, where $M_{A(B)}$ is the transformation matrix of PDC(DPC). Thus, the GMQD is given by

$$\begin{aligned} D^g(\rho(t)) &= \frac{1}{4} [(1-p)^4 c_1^2 + (1-p)^4 c_2^2 + r^2 + (1-p)^2 c_3^2] \\ &\quad - \max \left\{ (1-p)^4 c_1^2, (1-p)^4 c_2^2, r^2 + (1-p)^2 c_3^2 \right\} \end{aligned} \quad (16)$$

where we have assumed the parameter p are same in the two decoherence channels. Then it suffices to consider three separate cases.

Case 1. $r^2 + c_3^2 \geq \{c_1^2, c_2^2\}$. Then

$$\begin{aligned} D^g(\rho(t)) &= \frac{1}{4} [(1-p)^4 c_1^2 + (1-p)^4 c_2^2] \\ &= 2D^g[(\$_{PDC} \otimes \$_{DPC}) (|\beta_i\rangle \langle \beta_i|)] D^g(\rho(0)) \end{aligned} \quad (17)$$

Case 2. $c_1^2 \geq c_2^2 \geq r^2 + c_3^2$. Suppose p'_0 satisfies $(1 - p'_0)^4 c_1^2 = r^2 + (1 - p'_0)^2 c_3^2$. If $p \leq p'_0$, we have

$$\begin{aligned} D^g(\rho(t)) &= \frac{1}{4} [(1-p)^4 c_2^2 + r^2 + (1-p)^2 c_3^2] \\ &\geq \frac{1}{4} [(1-p)^4 c_2^2 + (1-p)^4 r^2 + (1-p)^4 c_3^2] \\ &= 2D^g[(\$_{PDC} \otimes \$_{DPC}) (|\beta_i\rangle \langle \beta_i|)] D^g(\rho(0)) \end{aligned} \quad (18)$$

If $p > p'_0$, we obtain

$$\begin{aligned} D^g(\rho(t)) &= \frac{1}{4} [(1-p)^4 c_1^2 + (1-p)^4 c_2^2] \\ &\geq \frac{1}{4} (1-p)^4 [c_2^2 + r^2 + c_3^2] \\ &= 2D^g[(\$_{PDC} \otimes \$_{DPC}) (|\beta_i\rangle \langle \beta_i|)] D^g(\rho(0)) \end{aligned} \quad (19)$$

Case 3. $c_1^2 \geq r^2 + c_3^2 \geq c_2^2$. In this case, the proof is similar to case 2.

One can also directly verify that the above relation holds for the other cases. ■

The theorem above provides us a method to compute the lower bound of the time evolution of a class of X states under two typical kinds of decoherence channels, without resort to the time evolution of the underlying quantum state itself. This inequality also holds for the one-sided PDC or DPC, which is summarized as the following Corollary:

Corollary 1. For the class of X states defined in Eq.(3), with one qubit being subject to PDC or DPC, we have

$$D^g(\rho(t)) \geq 2D^g[(\$_i \otimes I) (|\beta_i\rangle \langle \beta_i|)] D^g(\rho(0)) \quad (20)$$

Moreover, for $r = s = 0$, *i.e.* Bell diagonal states. Using Eq.(12), one can directly calculate the above inequality becomes equality which we summarize as follows:

Corollary 2. For arbitrary Bell-diagonal states subject to two DPCs, the evolution of GMQD is given by

$$D_{DPC}^g(\rho(t)) = 2D^g[(\$_{DPC} \otimes \$_{DPC}) |\beta_i\rangle \langle \beta_i|] D^g(\rho(0)) \quad (21)$$

where the two DPCs may be different. By far we have considered the time evolution of GMQD in the presence of two typical kinds of local decoherence noise. Another important decoherence noise is the amplitude damping channel(ADC). There are evidences to show that the above relation also hold for two identical ADCs, *i.e.*

$$D^g(\rho(t)) \geq 2D^g[(\$_{ADC} \otimes \$_{ADC}) (|\beta_i\rangle \langle \beta_i|)] D^g(\rho(0)) \quad (22)$$

In Fig.4, we plot the evolution of GMQD under two identical ADCs and its lower bound in Eq.(22) when (a) $c_1 = 0.1, c_2 = 0.1, c_3 = 0.2, r = s = 0.3$; (b) $c_1 = 0.2, c_2 = 0.05, c_3 = 0.3, r = 0.4, s = 0.1$.

IV. DISCUSSIONS AND CONCLUSIONS

In this work, we have investigated the level surfaces of GMQD for a class of two-qubit X states from the geometric picture. First, we plot the physical region for a class of two-qubit X states with fixed local Bloch vectors. It is shown that physical regions of the state have different geometry with the Bell-diagonal states and shrink with larger Bloch vectors. Second, the geometric picture is depicted in terms of the constant concurrence and GMQD, respectively. We find that the shape of the surfaces has close relationship with the value of GMQD and local Bloch vectors. Finally, we also investigate the dynamics of GMQD under two typical kinds of decoherence channels and obtain analytic results of the evolution of GMQD. It is shown that there exist a class of initial states for which the GMQD is not destroyed by decoherence in a finite time interval. Moreover, a direct factorization relationship between the initial and final GMQD subject to two typical kinds of decoherence channels is derived. This factorization law allows us to infer the evolution of entanglement under the influences of the environment without resort to the time evolution of the initial quantum state itself. An open question is whether this law holds under general local decoherence channels. Our results imply that further study on the dynamics of GMQD is required.

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Note added. After completing this manuscript, we became aware of an interesting related works by Yao Yao *et al*[60] recently.

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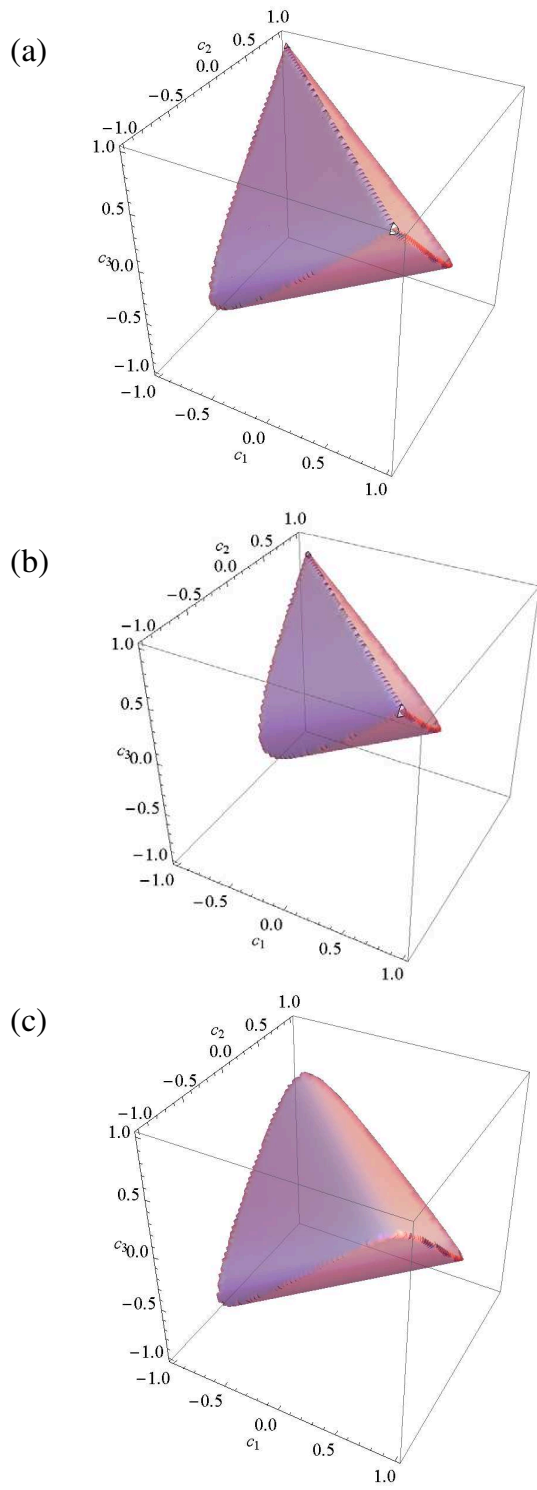


FIG. 1: (Color online). The geometry of the set of valid states with different r and s , respectively. (a) $r=s=0.3$, (b) $r=s=0.5$, (c) $r=0.4, s=0.1$

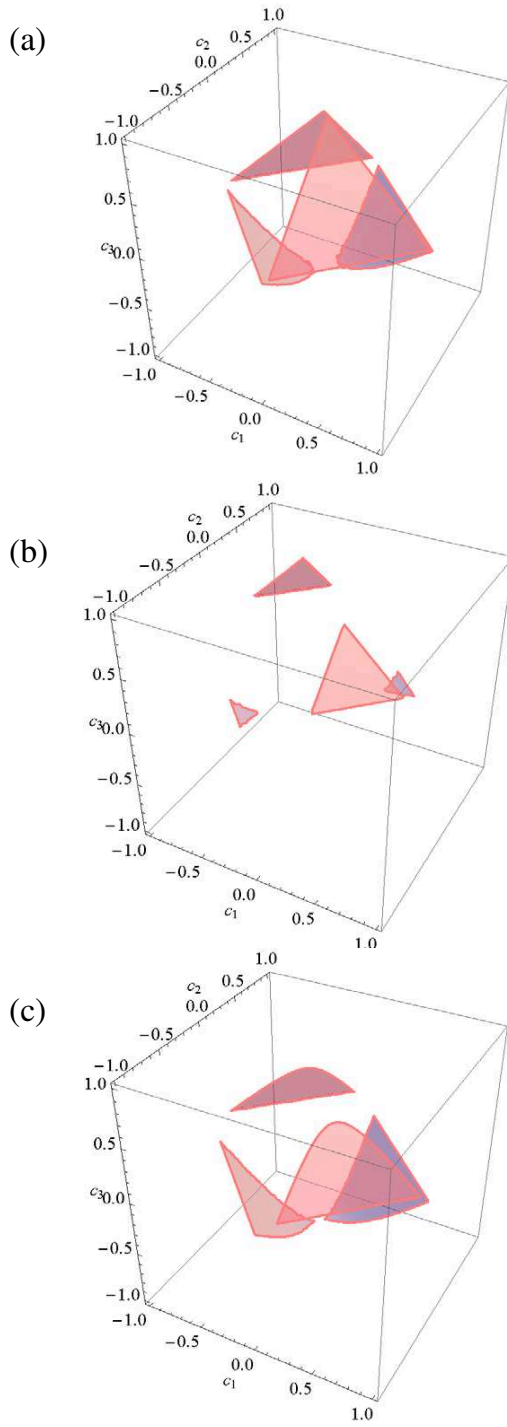


FIG. 2: (Color online). Surfaces of constant concurrence. (a) $r=s=0.3, C(\rho)=0.03$, (b) $r=s=0.5, C(\rho)=0.35$, (c) $r=0.4, s=0.1, C(\rho)=0.03$

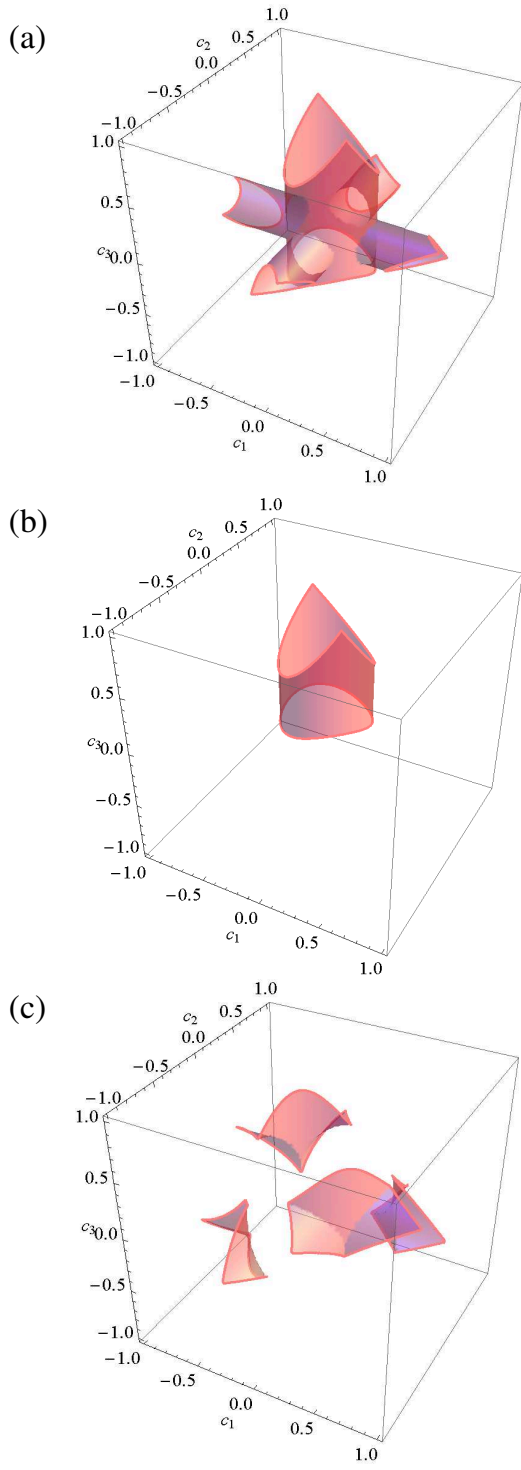


FIG. 3: (Color online). Surfaces of constant GMQD. (a) $r=s=0.3, D(\rho)=0.03$, (b) $r=s=0.5, D(\rho)=0.35$, (c) $r=0.4, s=0.1, D(\rho)=0.08$

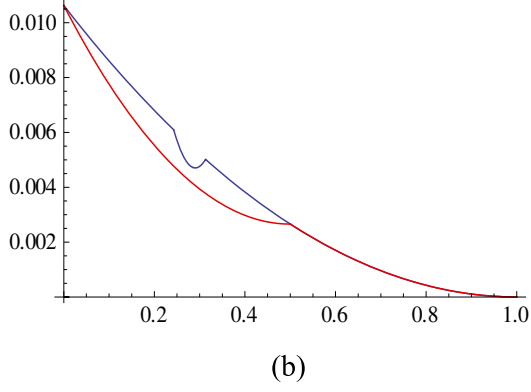
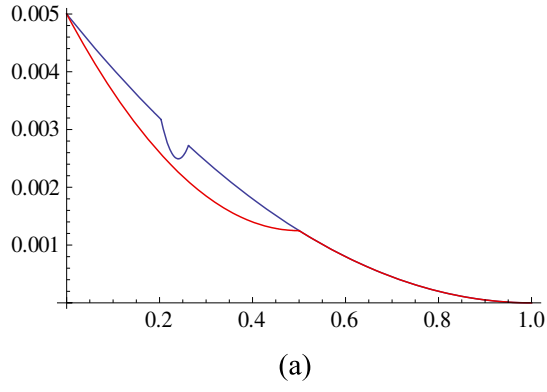


FIG. 4: (Color online). Plots of the dynamics of GMQD under two identical ADCs (blue line) and its lower bound defined in Eq. (22) (red line). (a) $c_1 = 0.1, c_2 = 0.1, c_3 = 0.2, r = s = 0.3$; (b) $c_1 = 0.2, c_2 = 0.05, c_3 = 0.3, r = 0.4, s = 0.1$.